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The uniqueness of the solution for the definite problem of a parabolic variational inequality

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Abstract

The uniqueness of the solution for the definite problem of a parabolic variational inequality is proved. The problem comes from the study of the optimal exercise strategies for the perpetual executive stock options with unrestricted exercise in financial market. Because the variational inequality is degenerate and the obstacle condition contains the partial derivative of an unknown function, it makes the theoretical study of the definite problem of the variational inequality problem very difficult. Firstly, the property which the value function satisfies is derived by applying the Jensen inequality. Then the uniqueness of the solution is proved by using this property and maximum principles.

Keywords: parabolic variational inequality; definite problem; uniqueness; Jensen inequality; maximum principles

1 Introduction

In this paper, the uniqueness of the solution of a parabolic variational inequality problem

$$\min\{\mathcal{L}u, \mathcal{B}u\} = 0, \quad u(z,0) = -1, (z,\tau) \in Q_\infty, \tag{1}$$

i.e.

$$\begin{aligned} &\mathcal{L}u \geq 0, \quad (z,\tau) \in Q_\infty, \\ &\mathcal{B}u \geq 0, \quad (z,\tau) \in Q_\infty, \\ &\mathcal{L}u \cdot \mathcal{B}u = 0, \quad (z,\tau) \in Q_\infty, \\ &u(z,0) = -1, \quad -\infty < z < +\infty, \end{aligned}$$

is studied. Here

$$\begin{aligned} &\mathcal{L}u \equiv r\tau \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2} - \mu \frac{\partial u}{\partial z}, \\ &\mathcal{B}u \equiv \frac{\partial u}{\partial \tau} + \left(e^z - K\right)^+ u, \\ &Q_\infty = (-\infty, +\infty) \times (0, A]. \end{aligned} \tag{2}$$

The following is a brief introduction of the financial background and related research of the definite solution problem (1)-(2).

Executive stock options are called ESOs for short, which are American call options that the company granted to the managers as a compensation. The underlying assets of ESOs are the company's stocks. The exercise of ESOs can be divided into block exercise and unrestricted exercise. Block exercise is identical to the exercise of the standard American call options, that is, the holders can exercise none of ESOs or exercise all of ESOs at any exercise time. In the unrestricted exercise situation, the holders can exercise any copies of ESOs at arbitrary exercise moment.

For the perpetual ESOs with unrestricted exercise, the total wealth utility maximization of the manager can be given by the following stochastic optimal control problem [1, 2]:

$$v(s,x,a) \equiv \sup_{m_\rho \in \mathcal{M}_0} E\left[U\left(x + \int_0^{+\infty} e^{-r\rho} (S_\rho - K)^+ \, dm_\rho \right) \Big| m_{0^-} = M - a, S_0 = s \right], \tag{3}$$

where E is the expectation under the market measure, M is the total number of ESOs held by the manager, and $\mathcal{M}_0$ is defined by the following:

$$\mathcal{M}_0 \equiv \left\{ m_\rho \,\middle|\, \begin{array}{l} m_\rho \textit{ is a non-negative, right continuous, increasing} \\ \textit{and adapted process, } \rho \in [0^-, +\infty),\ m_\infty = M \end{array} \right\}.$$

In addition, the meaning of each variable is as follows:

a is the number of ESOs held by the manager at time 0 before exercise, $0 < a \leq M$;
m_t is the number of ESOs exercised by the manager until time t;
S_t is the stock price at time t, and s is the stock price at time 0, $0 < s < +\infty$;
K is the strike price of ESOs, and r is the risk-free rate, where r is a constant and $r > 0$;
x is the manager's wealth at time 0 before exercise, $0 \leq x < +\infty$;
$U(\cdot)$ is the utility function of the manager, which is concave and increasing.

Suppose stock price S_t follows a geometric Brown motion in market measure,

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t,$$

where W_t is a standard Brown motion, α, σ are both constants, $0 < \alpha < r$, $\sigma > 0$.

Remark 1 The reason of taking $\alpha < r$ is that stocks with dividends are considered for the American call stock options in general. If the stocks do not have dividends, the American call options are actually European call options. ESOs are American call options, so the case of stocks with dividends is considered. Thus we take $\alpha < r$.

By stochastic optimal control theory [3, 4], $v(s,x,a)$ is the solution of the following parabolic variational inequalities:

$$\min\{\mathrm{L}v, \mathrm{B}v\} = 0, \quad 0 \leq x < +\infty, 0 < s < +\infty, 0 < a \leq M,$$

where

$$\begin{cases} \mathrm{L}v = ra\frac{\partial v}{\partial a} - \frac{\sigma^2}{2}s^2\frac{\partial^2 v}{\partial s^2} - \alpha s\frac{\partial v}{\partial s}, \\ \mathrm{B}v = \frac{\partial v}{\partial a} - (s-K)^+\frac{\partial v}{\partial x}, \end{cases}$$

and the condition of a definite solution is

$$v(s,x,0) = U(x).$$

Take the utility function $U(y) = -e^{-\gamma y}$ ($y \geq 0$), where the constant $\gamma > 0$ is the risk aversion coefficient, and let

$$\tau = \gamma a, \qquad s = e^z, \qquad v(s,x,a) = e^{-\gamma x} u(z,\tau). \tag{4}$$

Then $u(z,\tau)$ satisfies the following parabolic variational inequalities:

$$\min\left\{ r\tau \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2} - \mu \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \tau} + \left(e^z - K\right)^+ u \right\} = 0,$$

where $\mu = \alpha - \frac{\sigma^2}{2}$, $0 < \tau < +\infty$, $-\infty < z < +\infty$, and the condition of a definite solution becomes

$$u(z,0) = -1.$$

Because $0 < a \leq M$, $\gamma > 0$ are both constants, we only consider the case of $0 < \tau \leq A$ for arbitrary positive number A. Let $Q_\infty = (-\infty, +\infty) \times (0, A]$, we get the definite solution problem (1)-(2).

In [1], a permanent ESOs model with unrestrained exercise is established, and the definite solution problem (1)-(2) of a parabolic variational inequality is obtained. The existence and regularity of the solution to the definite solution problem (1)-(2) are proved in [2]. In [5], the properties of the free boundary for (1)-(2) are studied by using numerical method. However, the uniqueness of the solution to (1)-(2) has not yet been proved, which is the main content of this paper.

2 Proof

In order to prove the uniqueness of the solution to (1)-(2), we need the following two lemmas.

Lemma 1

$$\sup_{m_\rho \in \mathcal{M}_0} E\left[\int_0^{+\infty} e^{-r\rho}(S_\rho - K)^+ \, dm_\rho \,\middle|\, m_{0^-} = M - a, S_0 = s \right] = a V_\infty(s), \tag{5}$$

where $\mathcal{M}_0$ is the same as above, and $V_\infty(s)$ is the price of a standard perpetual American call option, that is, [6, 7]

$$\text{if } r = \alpha, \quad V_\infty(s) = s;$$

$$\text{if } r > \alpha, \quad V_\infty(s) = \begin{cases} (\frac{s}{S^*})^{\lambda_+}(S^* - K), & 0 < s \leq S^*, \\ s - K, & s > S^*, \end{cases}$$

where

$$s = e^z, \qquad \lambda_+ = \frac{-\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}, \qquad S^* = \frac{\lambda_+ K}{\lambda_+ - 1} > K,$$

S^* is the optimal exercise boundary of the standard perpetual American call options.

Proof By Remark 3 in [8], when the utility function is taken as $U(y) = y$ (that is, without utility function), the value function in the case of an unrestricted exercise is equal to the one in the case of a block exercise. That is, when there is no utility function, unrestricted exercise is equivalent to block exercise. Thus

$$\begin{aligned} &\sup_{m_\rho \in \mathcal{M}_0} E\left[\int_0^{+\infty} e^{-r\rho}(S_\rho - K)^+ \, dm_\rho \middle| m_{0^-} = M - a, S_0 = s\right] \\ &\quad = \sup_{\rho \in \mathcal{F}_0} E\left[ae^{-r\rho}(S_\rho - K)^+ | S_0 = s\right] \\ &\quad = aV_\infty(s), \end{aligned}$$

where $\mathcal{F}_0$ is defined by the following expression:

$$\mathcal{F}_0 \equiv \left\{\rho | \rho \in [0, +\infty) \text{ is a stopping time}\right\}.$$

This completes the proof. □

Lemma 2 *Suppose $v(s,x,a)$ is provided by* (3), *and $u(Z,\tau)$ is given by* (4), *then*

$$-1 \le u(z,\tau) \le -e^{-\tau V_\infty(e^z)}, \quad (z,\tau) \in Q_\infty, \tag{6}$$

where $Q_\infty = (-\infty, +\infty) \times (0, A]$, $s = e^z$, $V_\infty(s) = V_\infty(e^z)$ represents the price of a standard perpetual American call option, the specific expression of which is given by Lemma 1.

Proof By (3) and $U(y) = -e^{-\gamma y}$ $(y \ge 0)$, it is clear that $v(s,x,z) \ge -e^{-\gamma x}$.

By (4), we have $u(z,\tau) \ge -1$.

By (3), we obtain

$$\begin{aligned} v(s,x,a) &\le \sup_{m_\rho \in \mathcal{M}_0} U\left(E\left[x + \int_0^{+\infty} e^{-r\rho}(S_\rho - K)^+ \, dm_\rho \middle| m_{0^-} = M - a, S_0 = s\right]\right) \\ &\le U\left(\sup_{m_\rho \in \mathcal{M}_0} E\left[x + \int_0^{+\infty} e^{-r\rho}(S_\rho - K)^+ \, dm_\rho \middle| m_{0^-} = M - a, S_0 = s\right]\right) \\ &= U\big(x + aV_\infty(s)\big) = U\big(x + aV_\infty\big(e^z\big)\big), \end{aligned}$$

where the Jensen inequality is used in the first inequality, and equation (5) in Lemma 1 is applied in the second inequality. $V_\infty(e^z)$ is the price of a standard perpetual American call option.

By (4) and $U(y) = -e^{-\gamma y}$ $(y \ge 0)$, we get

$$u(z,\tau) \le -e^{-\gamma a V_\infty(e^z)} = -e^{-\tau V_\infty(e^z)}.$$

This completes the proof. □

Now to prove the uniqueness of the solution to the definite solution problem (1)-(2) which satisfies (6).

The definite solution problem (1)-(2) which satisfies (6) is equivalent to the following variational problem:

$$\mathcal{L}u \equiv r\tau\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2} - \mu\frac{\partial u}{\partial z} \geq 0, \quad (z,\tau) \in Q_\infty, \tag{7}$$

$$\mathcal{B}u \equiv \frac{\partial u}{\partial \tau} + \left(e^z - K\right)^+ u \geq 0, \quad (z,\tau) \in Q_\infty, \tag{8}$$

$$\mathcal{L}u \cdot \mathcal{B}u = 0, \quad (z,\tau) \in Q_\infty, \tag{9}$$

$$u(z,0) = -1, \quad -\infty < z < +\infty, \tag{10}$$

$$-1 \leq u(z,\tau) \leq -e^{-\tau V_\infty(e^z)}, \quad (z,\tau) \in Q_\infty. \tag{11}$$

Remark 2 For the continuous-time optimization problem, the corresponding variational inequality is usually obtained by formal derivation (not strictly derived). Is the solution of the variational inequality really the value function of the original problem? It needs to be further verified, that is, one has to prove a so-called 'verification theorem'. By a verification theorem, we can prove that when it is sufficiently smooth, the solution of the variational inequality (7)-(11) is equal to the value function of the corresponding singular stochastic control problem (obtained by an identical deformation of (3)).

Remark 3 Using the standard method in [4], we can also prove that the value function of the corresponding singular stochastic control problem (obtained by identical deformation of (3)) is a viscosity solution of the variational inequality (7)-(11).

Theorem 1 *Assume that* $r > \alpha$ *and the parameters* $A, r, \delta, \sigma, \gamma, K$ *are all positive constants. For any* $\rho > 0$ *and any* $\varepsilon \in (0,A)$, *let* $Q_\infty = (-\infty,+\infty) \times (0,A]$, $Q_\rho = (-\rho,\rho) \times (0,A]$, *and* $Q_\varepsilon\rho = (-\rho,\rho) \times [\varepsilon,A]$. *Then the problem* (7)-(11) *has a unique solution* $u(z,\tau)$ *satisfying*

$$u(z,\tau) \in W_\infty^{2,1}(Q_\rho) \cap C(\overline{Q}_\infty),$$

$$u_\tau \in C^{\frac{1}{2},\frac{1}{4}}(Q_{\varepsilon\rho}), \qquad u_{z\tau} \in L^2(Q_\rho), \qquad u_{\tau\tau} \in L^2(Q_\varepsilon\rho).$$

Proof Suppose there are two solutions u_1 and u_2 of (7)-(11) in $W^{2,1}_{\infty,\mathrm{loc}}(Q_\infty) \cap C(\overline{Q}_\infty)$. Denote $w = u_1 - u_2$. In order to prove $w = 0$, we need to first prove $w \leq 0$ in $\overline{Q}_\infty$.

If $z \to +\infty$, *i.e.* $s \to +\infty$, by (3) and $U(y) = -e^{-\gamma y}$, we have $v(s,x,a) \to 0$. By (4), we get $u(z,\tau) \to 0$, so $w(z,\tau) \to 0$ holds.

If $z \to -\infty$, by (11) and the expression of $V_\infty(e^z)$, we have $u(z,\tau) \to -1$, then $w(z,\tau) \to 0$ also holds.

Therefore

$$\lim_{z\to\infty} w(z,\tau) = 0, \quad \text{uniformly convergent for } \tau \in [0,A]. \tag{12}$$

By (12), $\forall\varepsilon > 0$, $\exists$ a sufficiently large $R_\varepsilon > 0$, *s.t.*

$$\left|w(z,\tau)\right| \leq \varepsilon, \quad |z| \geq R_\varepsilon, 0 \leq \tau \leq A. \tag{13}$$

By (10), we get

$$w(z,0)=0,\quad -\infty<z<\infty. \tag{14}$$

Set

$$\mathcal{O}_\varepsilon=(-R_\varepsilon,R_\varepsilon)\times(0,A]. \tag{15}$$

Suppose (z_*,τ_*) is a maximum point of $w(z,\tau)$ in $\mathcal{O}_\varepsilon$, then

$$\frac{\partial w}{\partial\tau}(z_*,\tau_*)\geq 0. \tag{16}$$

Now we prove

$$w(z_*,\tau_*)\leq\varepsilon. \tag{17}$$

(1) If $(z_*,\tau_*)\in\partial_p\mathcal{O}_\varepsilon$ ($\partial_p\mathcal{O}_\varepsilon$ is the parabolic boundary of $\mathcal{O}_\varepsilon$), then by (13)-(14), (17) holds.
(2) If $(z_*,\tau_*)\in\mathcal{O}_\varepsilon$ and $\mathcal{B}u_1(z_*,\tau_*)=0$, then by (8), we have

$$\mathcal{B}w(z_*,\tau_*)=\mathcal{B}(u_1-u_2)(z_*,\tau_*)\leq 0. \tag{18}$$

If $\mathcal{B}u_1(z_*,\tau_*)=0$, then (z_*,τ_*) belongs to stopping region (or exercise region), so $z_*>\ln K$, *i.e.*, $(e^{z*}-K)^+\neq 0$. By (8), (16), and (18), we obtain

$$w(z_*,\tau_*)=(u_1-u_2)(z_*,\tau_*)\leq-\frac{1}{(e^{z_*}-K)^+}\frac{\partial w}{\partial\tau}(z_*,\tau_*)\leq 0.$$

Thus $w(z_*,\tau_*)\leq\varepsilon$, *i.e.*, (17) holds.
(3) If $(z_*,\tau_*)\in\mathcal{O}_\varepsilon$ and $\mathcal{B}u_1(z_*,\tau_*)>0$, let

$$\widehat{\mathcal{O}}_\varepsilon=\left\{(z,\tau)\in\mathcal{O}_\varepsilon|\mathcal{B}u_1(z,\tau)>0\right\}.$$

Then $\widehat{\mathcal{O}}_\varepsilon$ is an open set of $\mathcal{O}_\varepsilon$, and $(z_*,\tau_*)\in\widehat{\mathcal{O}}_\varepsilon$. By (7)-(9), we get

$$\mathcal{L}u_1(z,\tau)=0\quad\text{in }\widehat{\mathcal{O}}_\varepsilon\subseteq Q_\infty,\qquad \mathcal{L}u_2(z,\tau)\geq 0\quad\text{in }\widehat{\mathcal{O}}_\varepsilon\subseteq Q_\infty.$$

Therefore

$$\mathcal{L}w=\mathcal{L}u_1-\mathcal{L}u_2=r\tau\frac{\partial w}{\partial\tau}-\frac{\sigma^2}{2}\frac{\partial^2 w}{\partial z^2}-\mu\frac{\partial w}{\partial z}\leq 0,\quad (z,\tau)\in\widehat{\mathcal{O}}_\varepsilon.$$

On the parabolic boundary of $\widehat{\mathcal{O}}_\varepsilon$,

$$\partial_p\widehat{\mathcal{O}}_\varepsilon\equiv\partial\widehat{\mathcal{O}}_\varepsilon\cap\{\tau<A\},$$

we have

$$w\leq\varepsilon.$$

In fact, let (z_0, τ_0) be a maximum point of $w(z,\tau)$ on the parabolic boundary $\partial_p\widehat{\mathcal{O}}_\varepsilon$. Because

$$\widehat{\mathcal{O}}_\varepsilon = \{(z,\tau) \in \mathcal{O}_\varepsilon | \mathcal{B}u_1(z,\tau) > 0\} = (-R_\varepsilon, a) \times (0, A],$$

for some $a \in (-R_\varepsilon, R_\varepsilon)$, we have $\tau \in (0, A)$ on $\partial_p\widehat{\mathcal{O}}_\varepsilon \equiv \partial\widehat{\mathcal{O}}_\varepsilon \cap \{\tau < A\}$. Then

$$\frac{\partial w}{\partial \tau}(z_0, \tau_0) = 0.$$

$\mathcal{B}u_1(z,\tau) = 0$ on $\partial_p\widehat{\mathcal{O}}_\varepsilon$, so $\mathcal{B}u_1(z_0,\tau_0) = 0$. Similar to the proof of (2), $w(z_0,\tau_0) \le \varepsilon$. Thus $w \le \varepsilon$ on $\partial_p\widehat{\mathcal{O}}_\varepsilon$.

Then by the maximum principle, $w \le \varepsilon$ on $\widehat{\mathcal{O}}_\varepsilon$. By $(z_*, \tau_*) \in \widehat{\mathcal{O}}_\varepsilon$, (17) holds.

In summary, (17) holds, so $w \le \varepsilon$ on $\mathcal{O}_\varepsilon$. Combining with (13), we have $w \le \varepsilon$ on $\overline{Q}_\infty$. Let $\varepsilon \to 0$, we get $w \le 0$ on $\overline{Q}_\infty$. Similarly, we can prove $w \ge 0$ on $\overline{Q}_\infty$ by exchanging u_1 and u_2. Thus $w = 0$ on $\overline{Q}_\infty$, *i.e.*, $u_1 = u_2$ on $\overline{Q}_\infty$.

This completes the proof. □

3 Concluding remarks

In this paper, we prove the uniqueness of the solution for a parabolic variational inequality with the following form: to find $u(z,\tau)$, s.t.

$$\min\{\mathcal{L}u, \mathcal{B}u\} = 0, \quad u(z,0) = -1, (z,\tau) \in Q_\infty,$$

where

$$\begin{aligned}
&\mathcal{L}u \equiv r\tau\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2} - \mu\frac{\partial u}{\partial z},\\
&\mathcal{B}u \equiv \frac{\partial u}{\partial \tau} + \left(e^z - K\right)^+ u,\\
&Q_\infty = (-\infty, +\infty) \times (0, A].
\end{aligned}$$

The problem is derived from the study of the optimal exercise strategy of the perpetual ESOs with unrestricted exercise in financial market. Because the variational inequality is degenerate, and the obstacle condition contains the partial derivative of the unknown function, it is difficult to carry out the theoretical research for the definite problem of the variational inequality. Firstly, the inequality (6) which $u(Z,\tau)$ satisfies is derived from known conditions, mainly using the related conclusion in [8] and the Jensen inequality. Then the uniqueness of the solution is proved by (6) and the maximum principle.

The utility function chosen in this paper is an exponential function. However, the conclusion of this paper still holds for other types of utility functions and the proof method is similar.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WY drafted the manuscript. LS helped to draft the manuscript and revised it. All authors read and approved the final manuscript.

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References

1. Rogers, LCG, Scheinkman, J: Optimal exercise of executive stock options. Finance Stoch. **11**(3), 357-372 (2007)
2. Song, LP, Yu, WH: A parabolic variational inequality related to the perpetual American executive stock options. Nonlinear Anal., Theory Methods Appl. **74**(17), 6583-6600 (2011)
3. Øksendal, B: Stochastic Differential Equations, 5th edn. Springer, Berlin (2000)
4. Pham, H: Continuous-Time Stochastic Control and Optimization with Financial Application. Springer, Berlin (2009)
5. Song, LP: A free boundary problem coming from the perpetual American ESOs. Chin. J. Eng. Math. **31**(4), 511-520 (2014)
6. Jaillet, P, Lamberton, D, Lapeyre, B: Variational inequalities and pricing of American options. Acta Appl. Math. **21**(3), 263-289 (1990)
7. Jiang, LS: The Mathematical Models and Methods in Option Pricing. Higher Education Press, Beijing (2003)
8. Song, LP, Yu, WH: The equivalence between block exercise and unrestricted exercise of executive stock options. J. Syst. Sci. Math. Sci. **36**(10), 1710-1720 (2016)